1. The concept of limit

Example 1.1. Let \( f(x) = \frac{x^2 - 4}{x - 2} \). Examine the behavior of \( f(x) \) as \( x \) approaches 2.

Solution. Let us compute some values of \( f(x) \) for \( x \) close to 2, as in the tables below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \frac{x^2 - 4}{x - 2} )</th>
<th>( x )</th>
<th>( f(x) = \frac{x^2 - 4}{x - 2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>3.9</td>
<td>2.1</td>
<td>4.1</td>
</tr>
<tr>
<td>1.99</td>
<td>3.99</td>
<td>2.01</td>
<td>4.01</td>
</tr>
<tr>
<td>1.999</td>
<td>3.999</td>
<td>2.001</td>
<td>4.001</td>
</tr>
<tr>
<td>1.9999</td>
<td>3.9999</td>
<td>2.0001</td>
<td>4.0001</td>
</tr>
</tbody>
</table>

We see from the first table that \( f(x) \) is getting closer and closer to 4 as \( x \) approaches 2 from the left side. We express this by saying that “the limit of \( f(x) \) as \( x \) approaches 2 from left is 4”, and write

\[
\lim_{x \to 2^-} f(x) = 4.
\]

Similarly, by looking at the second table, we say that “the limit of \( f(x) \) as \( x \) approaches 2 from right is 4”, and write

\[
\lim_{x \to 2^+} f(x) = 4.
\]

We call \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \) one-sided limits. Since the two one-sided limits of \( f(x) \) are the same, we can say that “the limit of \( f(x) \) as \( x \) approaches 2 is 4”, and write

\[
\lim_{x \to 2} f(x) = 4.
\]

Note that since \( x^2 - 4 = (x - 2)(x + 2) \), we can write

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4,
\]

where we can cancel the factors of \( (x - 2) \) since in the limit as \( x \to 2 \), \( x \) is close to 2, but \( x \neq 2 \), so that \( x - 2 \neq 0 \). Below, find the graph of \( f(x) \), from which it is also clear that \( \lim_{x \to 2} f(x) = 4 \).
Example 1.2. Let \( g(x) = \frac{x^2 - 5}{x - 2} \). Examine the behavior of \( g(x) \) as \( x \) approaches 2.

Solution. Based on the graph and tables of approximate function values shown below,

\[
\begin{array}{|c|c|}
\hline
x & g(x) = \frac{x^2 - 5}{x - 2} \\
\hline
1.9 & 13.9 \\
1.99 & 103.99 \\
1.999 & 1003.999 \\
1.9999 & 10,003.9999 \\
\hline
\end{array}
\]

observe that as \( x \) gets closer and closer to 2 from the left, \( g(x) \) increases without bound and as \( x \) gets closer and closer to 2 from the left, \( g(x) \) decreases without bound. We express this situation by saying that the limit of \( g(x) \) as \( x \) approaches 2 from the left is \( \infty \), and \( g(x) \) as \( x \) approaches 2 from the right is \( -\infty \) and write

\[
\lim_{x \to 2^-} g(x) = \infty, \quad \lim_{x \to 2^+} g(x) = -\infty.
\]

Since there is no common value for the one-sided limits of \( g(x) \), we say that the limit of \( g(x) \) as \( x \) approaches 2 does not exists and write

\[\lim_{x \to 2} g(x) \text{ does not exits.}\]

Example 1.3. Use the graph below to determine \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \), \( \lim_{x \to 1} f(x) \) and \( \lim_{x \to 1^-} f(x) \).

Solution. It is clear from the graph that

\[\lim_{x \to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = -1.\]
Since \( \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) \), \( \lim f(x) \) does not exist. It is also clear from the graph that
\[
\lim_{x \to 1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 1.
\]
Since \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \), \( \lim f(x) = 1 \).

**Example 1.4.**

1. Graph \( \frac{3x + 9}{x^2 - 9} \).
2. Evaluate \( \lim_{x \to -3} \frac{3x + 9}{x^2 - 9} \).
3. Evaluate \( \lim_{x \to 3} \frac{3x + 9}{x^2 - 9} \).

**Solution.**

1. Note that \( f(x) = \frac{3x + 9}{x^2 - 9} = \frac{3}{x - 3} \) for \( x \neq -3 \). Then, by shifting and scaling the graph of \( y = \frac{1}{x} \), we obtain

![Graph of \( y = \frac{1}{x} \)](image)

2. Since \( f(x) = \frac{3x + 9}{x^2 - 9} = \frac{3}{x - 3} \) for \( x \neq -3 \), \( \lim_{x \to -3} \frac{3x + 9}{x^2 - 9} = \lim_{x \to -3} \frac{3}{x - 3} = -\frac{1}{2} \).

3. It is seen from the graph that \( \lim_{x \to 3} \frac{3x + 9}{x^2 - 9} = \pm \infty \). Hence, \( \lim_{x \to 3} \frac{3x + 9}{x^2 - 9} \) does not exist.

**Example 1.5.** Evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \).

**Solution.** From the following tables and the graph

![Graph of \( \frac{\sin x}{x} \)](image)

one can conjecture that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

From now on, we will use the following fact without giving its proof.

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
Example 1.6. Evaluate \( \lim_{x \to 0} \frac{x}{|x|} \).

Solution. Note that
\[
\frac{x}{|x|} = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0
\end{cases}
\]
So, \( \lim_{x \to 0^+} \frac{x}{|x|} = 1 \) while \( \lim_{x \to 0^-} \frac{x}{|x|} = -1 \). Since the left limit is not equal to the right limit, \( \lim_{x \to 0} \frac{x}{|x|} \) does not exist.

Example 1.7. Sketch the graph of \( f(x) = \begin{cases} 
2x & \text{if } x < 2 \\
x^2 & \text{if } x \geq 2
\end{cases} \) and identify each limit.

(a) \( \lim_{x \to 2^-} f(x) \)
(b) \( \lim_{x \to 2^+} f(x) \)
(c) \( \lim_{x \to 2} f(x) \)
(d) \( \lim_{x \to 1} f(x) \)

Solution.
The graph is shown below.

And,
(a) \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x = 4 \)
(b) \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4 \)
(c) \( \lim_{x \to 2} f(x) = 4 \)
(d) \( \lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2 \)

Example 1.8. Sketch the graph of \( f(x) = \begin{cases} 
x^3 - 1 & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
\sqrt{x} + 1 - 2 & \text{if } x > 0
\end{cases} \) and identify each limit.

(a) \( \lim_{x \to 0^-} f(x) \)
(b) \( \lim_{x \to 0^+} f(x) \)
(c) \( \lim_{x \to 0} f(x) \)
(d) \( \lim_{x \to 1} f(x) \)
Solution.

The graph is shown below.

And,

(a) \( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x^3 - 1) = -1 \)

(b) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x + 1} - 2 = -1 \)

(c) \( \lim_{x \to 0} f(x) = -1 \)

(d) \( \lim_{x \to -3} f(x) = \lim_{x \to -3} (x^3 - 1) = -2 \)

(e) \( \lim_{x \to 3} f(x) = \lim_{x \to 3} \sqrt{x + 1} - 2 = 0 \)

2. Computation of Limits

It is easy to see that for any constant \( c \) and any real number \( a \),

\[ \lim_{x \to a} c = c, \]

and

\[ \lim_{x \to a} x = a. \]

The following theorem lists some basic rules for dealing with common limit problems

**Theorem 2.1** Suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist and let \( c \) be any constant. Then,

(i) \( \lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x) \),

(ii) \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \),

(iii) \( \lim_{x \to a} [f(x) g(x)] = \left[ \lim_{x \to a} f(x) \right] \left[ \lim_{x \to a} g(x) \right], \) and

(iv) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) provided \( \lim_{x \to a} g(x) \neq 0 \).

By using (iii) of Theorem 2.1, whenever \( \lim_{x \to a} f(x) \) exits,

\[ \lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x) f(x)] = \left[ \lim_{x \to a} f(x) \right] \left[ \lim_{x \to a} f(x) \right] = \left[ \lim_{x \to a} f(x) \right]^2. \]

Repeating this argument, we get that

\[ \lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n, \]

for any positive integer \( n \). In particular, for any positive integer \( n \) and any real number \( a \),

\[ \lim_{x \to a} x^n = a^n. \]

**Example 2.1.** Evaluate

(1) \( \lim_{x \to -2} (3x^2 - 5x + 4). \)
(2) \( \lim_{x \to 3} \frac{x^3 - 5x + 4}{x^2 - 2} \).

(3) \( \lim_{x \to 1} \frac{x^2 - 1}{1 - x} \).

**Theorem 2.2** For any polynomial \( p(x) \) and any real number \( a \),

\[
\lim_{x \to a} p(x) = p(a).
\]

**Theorem 2.3** Suppose that \( \lim_{x \to a} f(x) = L \) and \( n \) is any positive integer. Then,

\[
\lim_{x \to a} n \sqrt[n]{f(x)} = n \sqrt[n]{\lim_{x \to a} f(x)} = n \sqrt[n]{L},
\]

where for \( n \) even, we assume that \( L > 0 \).

**Example 2.2.** Evaluate

(1) \( \lim_{x \to 2} \sqrt[3]{3x^2 - 2x} \).

(2) \( \lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} \).

**Theorem 2.4** For any real number \( a \), we have

(i) \( \lim_{x \to a} \sin x = \sin a \),

(ii) \( \lim_{x \to a} \cos x = \cos a \),

(iii) \( \lim_{x \to a} e^x = e^a \),

(iv) \( \lim_{x \to a} \ln x = \ln a \), for \( a > 0 \),

(v) \( \lim_{x \to a} \sin^{-1} x = \sin^{-1} a \), for \(-1 < a < 1\),

(vi) \( \lim_{x \to a} \cos^{-1} x = \cos^{-1} a \), for \(-1 < a < 1\),

(vii) \( \lim_{x \to a} \tan^{-1} x = \tan^{-1} a \), for \(-\infty < a < \infty\),

(viii) if \( p \) is a polynomial and \( \lim_{x \to p(a)} f(x) = L \), then \( \lim_{x \to a} f(p(x)) = L \).

**Example 2.3.** Evaluate \( \lim_{x \to 0} \sin^{-1} \left( \frac{x + 1}{2} \right) \).

**Example 2.4.** Evaluate \( \lim_{x \to 0} (x \cot x) \).

**Theorem 2.5** (Sandwich Theorem) Suppose that

\[
f(x) \leq g(x) \leq h(x)
\]

for all \( x \) in some interval \((c, d)\), except possibly at the point \( a \in (c, d) \) and that

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,
\]

for some number \( L \). Then, it follows that

\[
\lim_{x \to a} g(x) = L, \text{ too.}
\]
Example 2.5. Evaluate \( \lim_{x \to 0} \left[ x^2 \cos \left( \frac{1}{x} \right) \right] \).

Example 2.6. Evaluate \( \lim_{x \to 0} f(x) \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
  x^2 + 2 \cos x + 1 & \text{if } x < 0 \\
  e^x - 4 & \text{if } x \geq 0
\end{cases}
\]

Example 2.7. Evaluate.

1. \( \lim_{x \to 0} \frac{1 - e^{2x}}{1 - e^x} \).
2. \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x^3 - 1} \).
3. \( \lim_{x \to 0} \frac{\sin x}{\tan x} \).
4. \( \lim_{x \to 0} \frac{5x}{\tan 2x} \).
5. \( \lim_{x \to 0} \frac{x^2 + x}{xe^{-2x+1}} \).
6. \( \lim_{x \to 0} x^2 \csc^2 x \).
7. \( \lim_{x \to -1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) \).
8. \( \lim_{x \to 0} \frac{(1 + x)^3 - 1}{x} \).
9. \( \lim_{x \to 0} \frac{\sin |x|}{x} \).
10. \( \lim_{x \to 1} x \).
11. \( \lim_{x \to 1.5} [x] \).
12. \( \lim_{x \to -1} (x - [x]) \).

3. Continuity and its Consequences

A function \( f \) is **continuous** at \( x = a \) when

(i) \( f(a) \) is defined,
(ii) \( \lim_{x \to a} f(x) \) exists, and
(iii) \( \lim_{x \to a} f(x) = f(a) \).

Otherwise \( f \) is said to be **discontinuous** at \( x = a \).

Example 3.1. Let us see some examples of functions that are discontinuous at \( x = a \).

1. The function is not defined at \( x = a \). The graph has a hole at \( x = a \).
Example 3.2. Determine where \( f(x) = \frac{x^2 + 2x - 3}{x - 1} \) is continuous.

The point \( x = a \) is called a \textit{removable} discontinuity of a function \( f \) if one can remove the discontinuity by redefining the function at that point. Otherwise, it is called a \textit{nonremovable} or an \textit{essential} discontinuity of \( f \). Clearly, a function has a removable discontinuity at \( x = a \) if and only if \( \lim_{x \to a} f(x) \) exists and is finite.
Example 3.3. Classify all the discontinuities of

1. \( f(x) = \frac{x^2 + 2x - 3}{x - 1} \).
2. \( f(x) = \frac{1}{x^2} \).
3. \( f(x) = \cos \frac{1}{x} \).

Theorem 3.1 All polynomials are continuous everywhere. Additionally, \( \sin x \), \( \cos x \), \( \tan^{-1} x \) and \( e^x \) are continuous everywhere. \( \sqrt{x} \) is continuous for all \( x \), when \( n \) is odd and for \( x > 0 \), when \( n \) is even. We also have \( \ln x \) is continuous for \( x > 0 \) and \( \sin^{-1} x \) and \( \cos^{-1} x \) are continuous for \(-1 < x < 1 \).

Theorem 3.2 Suppose that \( f \) and \( g \) are continuous at \( x = a \). Then all of the following are true:

1. \( (f \pm g) \) is continuous at \( x = a \),
2. \( (f \cdot g) \) is continuous at \( x = a \), and
3. \( (f/g) \) is continuous at \( x = a \) if \( g(a) \neq 0 \).

Example 3.4. Find and classify all the discontinuities of \( \frac{x^4 - 3x^2 + 2}{x^2 - 3x - 4} \).

Theorem 3.3 Suppose that \( \lim_{x \to a} g(x) = L \) and \( f \) is continuous at \( L \). Then,

\[
\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(L).
\]

Corollary 3.4 Suppose that \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \). Then the composition \( f \circ g \) is continuous at \( a \).

Example 3.5. Determine where \( h(x) = \cos(x^2 - 5x + 2) \) is continuous.

If \( f \) is continuous at every point on an open interval \((a, b)\), we say that \( f \) is continuous on \((a, b)\).
We say that \( f \) is continuous on the closed interval \([a, b]\), if \( f \) is continuous on the open interval \((a, b)\) and

\[
\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).
\]

Finally, if \( f \) is continuous on all of \((-\infty, \infty)\), we simply say that \( f \) is continuous.

Example 3.6. Determine the interval(s) where \( f \) is continuous, for

1. \( f(x) = \sqrt{4 - x^2} \),
2. \( f(x) = \ln(x - 3) \).

Example 3.7. For what value of \( a \) is

\[
f(x) = \begin{cases} 
x^2 - 1, & x < 3 \\
2ax, & x \geq 3
\end{cases}
\]

continuous at every \( x \)?
Example 3.8. Let

\[ f(x) = \begin{cases} 
2 \text{sgn}(x-1), & x > 1, \\
a, & x = 1, \\
x + b, & x < 1.
\end{cases} \]

If \( f \) is continuous at \( x = 1 \), find \( a \) and \( b \).

**Theorem 3.5** (Intermediate Value Theorem) Suppose that \( f \) is continuous on the closed interval \([a, b]\) and \( W \) is any number between \( f(a) \) and \( f(b) \). Then, there is a number \( c \in [a, b] \) for which \( f(c) = W \).

**Example 3.9.** Two illustrations of the intermediate value theorem:

![Intermediate Value Theorem Illustration](image)

**Corollary 3.6** Suppose that \( f \) is continuous on \([a, b]\) and \( f(a) \) and \( f(b) \) have opposite signs. Then, there is at least one number \( c \in (a, b) \) for which \( f(c) = 0 \).

4. **Limits involving infinity; asymptotes**

If the values of \( f \) grow without bound, as \( x \) approaches \( a \), we say that \( \lim_{x \to a} f(x) = \infty \). Similarly, if the values of \( f \) become arbitrarily large and negative as \( x \) approaches \( a \), we say that \( \lim_{x \to a} f(x) = -\infty \).

A line \( x = a \) is a **vertical asymptote** of the graph of a function \( y = f(x) \) if either

\[ \lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty. \]

**Example 4.1.** Evaluate

1. \( \lim_{x \to -3} \frac{1}{x+3} \)
2. \( \lim_{x \to -3} \frac{1}{(x+3)^2} \)
3. \( \lim_{x \to 1} \frac{(x-2)^2}{x^2 - 4} \)
4. \( \lim_{x \to 2} \frac{x^2 - 4}{x-2} \)
5. \( \lim_{x \to -2} \frac{x^2 - 4}{x - 3} \)
(6) \( \lim_{x \to 2^-} \frac{x - 3}{x^2 - 4} \).
(7) \( \lim_{x \to 2} \frac{x - 3}{x^2 - 4} \).
(8) \( \lim_{x \to 2} \frac{2 - x}{(x - 2)^3} \).
(9) \( \lim_{x \to 5} \frac{1}{(x - 5)^3} \).
(10) \( \lim_{x \to -2} \frac{x + 1}{(x - 3)(x + 2)} \).
(11) \( \lim_{x \to \pi/2} \tan x \).

Intuitively, \( \lim_{x \to \infty} f(x) = L \) (or, \( \lim_{x \to -\infty} f(x) = L \)) if \( x \) moves increasingly far from the origin in the positive direction (or, in the negative direction), \( f(x) \) gets arbitrarily close to \( L \).

**Example 4.2.** Clearly, \( \lim_{x \to \infty} \frac{1}{x} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x} = 0 \).

A line \( y = b \) is a horizontal asymptote of the graph of a function \( y = f(x) \) if either \( \lim_{x \to \infty} f(x) = b \) or \( \lim_{x \to -\infty} f(x) = b \).

**Example 4.3.** Evaluate \( \lim_{x \to \infty} 5 + \frac{1}{x} \).

**Theorem 4.1** For any rational number \( t > 0 \),
\[
\lim_{x \to \pm \infty} \frac{1}{x^t} = 0,
\]
where for the case where \( x \to -\infty \), we assume that \( t = p/q \) where \( q \) is odd.

**Theorem 4.2** For any polynomial of degree \( n > 0 \), \( p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \), we have
\[
\lim_{x \to \infty} p_n(x) = \begin{cases} 
\infty & \text{if } a_n > 0 \\
-\infty & \text{if } a_n < 0
\end{cases}
\]

**Example 4.4.** Evaluate
\[
(1) \lim_{x \to \infty} \frac{2x^3 + 7}{3x^3 - x^2 + x + 7}.
(2) \lim_{x \to \infty} \frac{1}{x^2 - 7}.
(3) \lim_{x \to \infty} \frac{1}{x}.
(4) \lim_{x \to \infty} \frac{\sqrt{x^2 + 2x + 3}}{x} - x.
(5) \lim_{x \to \infty} \frac{\sin x}{x}.
(6) \lim_{x \to \infty} \sin x.
\]

**Example 4.5.** Find the horizontal asymptote(s) of \( f(x) = \frac{2 - x + \sin x}{x + \cos x} \).
Let \( f(x) = \frac{P(x)}{Q(x)} \). If (the degree of \( P \)) = (the degree of \( Q \))+1, then the graph of \( f \) has an oblique (slant) asymptote. We find an equation for the asymptote by dividing numerator by denominator to express \( f \) as a linear function plus a remainder that goes to 0 as \( x \to \pm \infty \).

**Example 4.6.** Find the asymptotes of the graph of \( f \), if

1. \( f(x) = \frac{x^2 - 3}{2x - 4} \).
2. \( f(x) = \frac{2x}{x + 1} \).

**Example 4.7.** Evaluate

1. \( \lim_{x \to 0^-} e^{\frac{1}{x}} \).
2. \( \lim_{x \to 0^+} e^{\frac{1}{x}} \).
3. \( \lim_{x \to \infty} \tan^{-1} x \).
4. \( \lim_{x \to -\infty} \tan^{-1} x \).
5. \( \lim_{x \to 0^+} \ln x \).
6. \( \lim_{x \to \infty} \ln x \).
7. \( \lim_{x \to 0} \sin \left( \frac{e^{-\frac{1}{x^2}}}{x} \right) \).
8. \( \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} \).
9. \( \lim_{x \to -\infty} \frac{\sqrt{x^2 + 1}}{x} \).